On Welded Tree Problems

Shrinidhi Tegnahally Sridhara

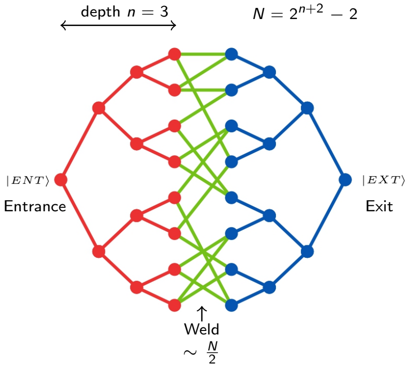
LaBRI, Université de Bordeaux

Based on joint work in progress with Yvan Le Borgne and Yassine Hamoudi

What are welded trees?

Definition

Welded trees are family of graphs G_n, obtained by joining the leaves of two complete binary trees of depth n using two random perfect matchings or a random cycle.



- A classical random walk resembles a binary tree at each vertex far from root.
- In the weld, the walker cannot determine the direction of exploration.
- Thus, a classical walker starting from $|ENT\rangle$ is lost in the weld, finding it difficult to reach $|EXT\rangle$.

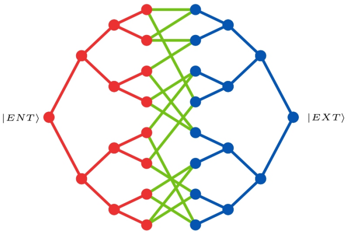
Why are they interesting?

- Welded trees are artificial graphs designed to exhibit the advantage of quantum walks over *any* classical algorithm [CCD$^+$03].

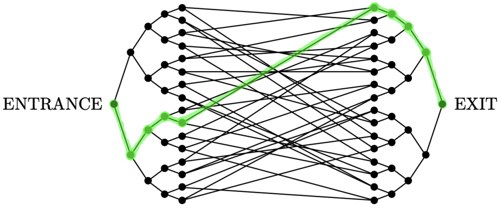
- They have been extensively studied in the following areas,
  - Quantum Walks - [JZ23], [Bel24], [LLL24].
  - Quantum Adiabatic Computation - [GHV21].
  - Graph Property testing - [BDCG$^+$24].

- For certain search problems, welded trees offer significant speedup over classical algorithms — ranging from polynomial to exponential.

Outline of the talk



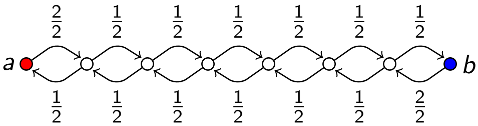
- Given the entrance vertex $|ENT\rangle$, find the exit vertex $|EXT\rangle$.



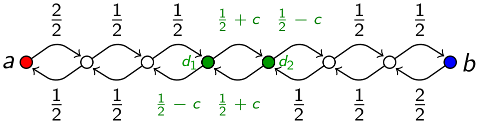
- An extension: Given the entrance vertex $|ENT\rangle$, find a *path* to the exit vertex $|EXT\rangle$.

Motivation to study quantum walks on welded trees

Line without defect (Classical random walk)

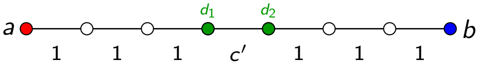


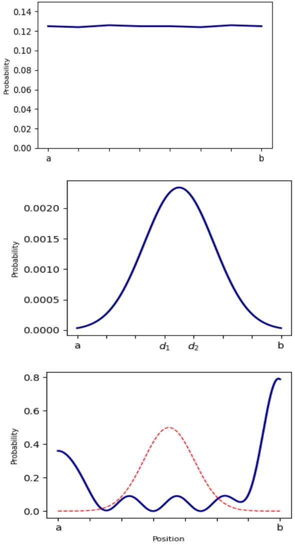
Line with defect (Classical random walk)



$\Updownarrow$

Line with defect (Quantum walk dynamics)





Problems on Welded Trees

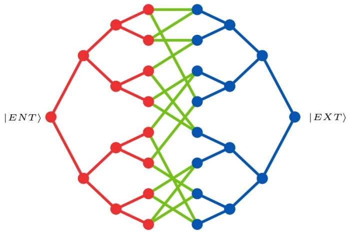
- We assume that the welded tree W is provided to us only via query access to an adjacency list oracle O_W that reveals the adjacent vertices of the vertex queried.
- $O_W \left|v\right\rangle \left|i\right\rangle \left|0\right\rangle \rightarrow \left|v\right\rangle \left|i\right\rangle \left|N_v(i)\right\rangle$, where $N_v(i)$ is the i^{th} neighbour of v.
- We are concerned with the query complexity i.e., the number of oracle queries made by the algorithm.
- The vertices of W are named by using random bit strings in $\{0, 1\}^{2n}$ which we call *id*s.[1]

[1] Notice that the number of *id*s is exponentially more than the number of vertices.

Search problems on Welded trees - 1

Definition

Finding the Exit Problem (FEP): Given the *id* of the entrance vertex of a welded tree $W \in G_n$ and the oracle O_W, return the *id* of the exit vertex of W.



- Local view of W + regularity + random labelling $\rightarrow$ Hard for a classical random walk.
- In [LBTS24], we provide exact analysis of the exponential time taken by a classical random walk, both non-backtracking and backtracking.

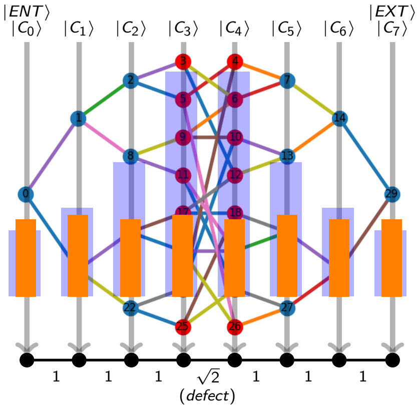
Search problems on Welded trees - 1

Definition

Finding the Exit Problem (FEP): Given the *id* of the entrance vertex of a welded tree $W \in G_n$ and the oracle O_W, return the *id* of the exit vertex of W.

- Childs et al. [CCD+03] showed that *any* classical algorithm must perform $\Omega(2^{n/6})$ queries before returning the exit with significant probability.
  - Reason: It is difficult to find cycles or the exit in W.
- In [LBTS24], we prove the folklore lower bound of $\Omega(2^{n/2})$ queries for any classical algorithm solving FEP.

Search problems on Welded trees - 1



$$\langle EXT|\, e^{-iHt}\, |ENT\rangle \geq \mathcal{O}(\tfrac{1}{n})$$
where $t = poly(n)$

- Childs et al. [CCD+03] show that a quantum walk-based algorithm finds the exit in $\mathcal{O}(n^4)$ calls to the oracle O_W.
- The speedup comes from viewing the quantum walk over welded trees as a walk on a line with a *defect*, which enables exponential speedup.

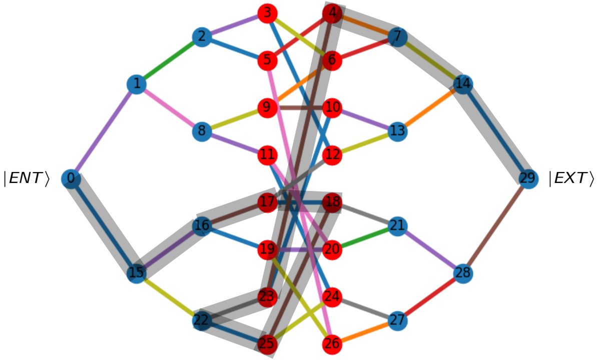
$$|C_j\rangle = \frac{1}{\sqrt{|C_j|}} \sum_{u \in C_j} |u\rangle\,,$$

$$H\,|C_j\rangle = \alpha\,|C_{j-1}\rangle + \beta\,|C_{j+1}\rangle$$

Search problems on Welded trees - 2

Definition

Finding a Path to Exit Problem (FPEP): Given the *id* of the entrance vertex of a welded tree $W \in G_n$ and the oracle O_W, return a path from the entrance vertex to the exit vertex of W.



Search problems on Welded trees - 2

- Aaronson proposed this extension of FEP and asked if FPEP could be solved in sub-exponentially ($2^{o(n)}$) many queries by a quantum algorithm [Aar21].
- The quantum walk algorithm efficiently finds the EXIT by exploring exponentially many paths in superposition, but *doesn't output* any of them.
- The intermediate state of a quantum algorithm cannot be recorded without collapsing the superposition and disrupting the algorithm.

Search problems on Welded trees - 2

- Any classical algorithm for FEP can be adapted to solve FPEP by tracking the path.
- There is a classical algorithm by Ben-David that solves FPEP (and FEP) in $\mathcal{O}(2^{n/2})$ queries, so the classical complexity of FPEP is $\Theta(2^{n/2})$ [(ps14].
- What if the quantum algorithm also remembers the vertices?
  - *Unlikely to succeed*

Search problems on Welded trees - 2

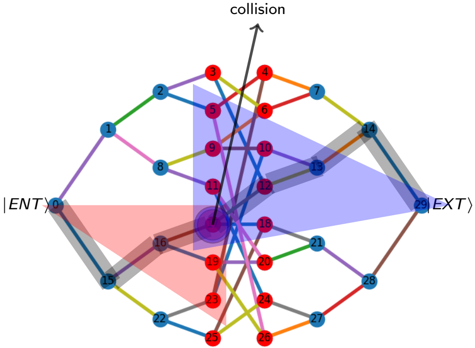
- [CCG22] showed that quantum algorithms that store paths from the root ($\mathcal{F}$) require exponentially many queries to solve FPEP; the power of quantum algorithms lies in forgetting the path!

- Their lower bound arises from the fact that algorithms in $\mathcal{F}$ can be classically simulated with polynomial queries and bounded error probability.

Search problems on Welded trees - 2

- We propose two algorithms that are not in $\mathcal{F}$.
- Both make use of the efficient quantum algorithm for FEP as a subroutine.
- The first algorithm is *mostly* classical and solves FPEP using $\tilde{\mathcal{O}}(2^{n/2})$ queries to oracle.
- The second one is a quantum algorithm and solves FPEP using $\tilde{\mathcal{O}}(2^{n/3})$ queries to oracle. A first speedup (polynomial) over the classical one!

Quantum Algorithm for FPEP - 1



- Find $|EXT\rangle$ using a quantum algorithm for FEP.
- Repeat:
  - $p_1 \leftarrow$ Sample a non-backtracking path of length n from $|ENT\rangle$.
  - $p_2 \leftarrow$ Sample a non-backtracking path of length $n+1$ from $|EXT\rangle$.
- Until $last(p_1) = last(p_2)$ (collision).

Quantum Algorithm for FPEP - 1

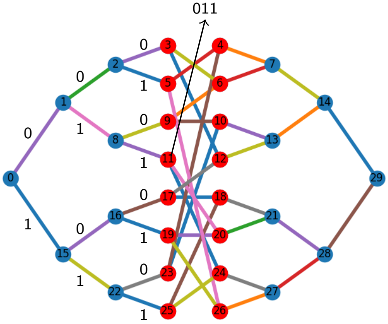
- Finding the exit can be done in $O(n)$ queries by using the optimal algorithm by Jefferey and Zur [JZ23].
- By *birthday paradox*, if $x = y = 2^{n/2}$, then there exists a collision with probability at least $\frac{1}{2}$.
- This requires $\tilde{\mathcal{O}}(2^{n/2})$ calls to the oracle O_W and $\mathcal{O}(2^{n/2})$ memory.
- This huge memory requirement can be reduced using cryptographic primitives to $\mathcal{O}(n^2)$. - *An exponential speedup in memory?*
- Total query complexity is $O(n) + \tilde{\mathcal{O}}(2^{n/2}) = \tilde{\mathcal{O}}(2^{n/2})$.

Quantum Algorithm for FPEP - 2

- The first algorithm made use of classical collision finding, can we do better by using quantum methods?

- We use the idea of quantum collision finding algorithm by Brassard et al. to get a polynomial speedup over any classical algorithm [BHT98].

- A non-backtracking walk from the roots to the weld can be uniquely represented by a bit-string.



Quantum Algorithm for FPEP - 2

- $T \leftarrow$ Sample $2^{n/3}$ non-backtracking paths of length $n+1$ from exit.
- Let $\mathcal{B}$ be the oracle such that

$$|\psi\rangle := \mathcal{B}\,|0\rangle\,|0\rangle \rightarrow \frac{1}{\sqrt{2^n}} \sum_{i \in \{0,1\}^n} (-1)^{f(i)}\,|i\rangle\,|P_i\rangle$$

where,

$$f(i) = \begin{cases} 1, & \text{last}(P_i) \in \text{last}(T) \\ 0, & \text{otherwise} \end{cases}$$

Quantum Algorithm for FPEP - 2

- Now, the task reduces to finding a path P_i marked by $f(i)$ in $|\psi\rangle$.
- Let $|G\rangle = \sum_{i:f(i)=1} |i\rangle |P_i\rangle$ be the set of *marked* paths.
- Initially $p = |\langle G|\psi\rangle|^2 = \frac{2^{n/3}}{2^n} = 2^{-2n/3}$.
- Using Grover's search, $O(\frac{1}{\sqrt{p}})$ applications of oracle $\mathcal{B}$ are enough to find a marked path with high probability [Gro96].
- Thus the query complexity of this algorithm is $\tilde{\mathcal{O}}(2^{n/3})$.

Summary of the results

We show the following results,

- A quantum algorithm that solves FPEP in $\tilde{\mathcal{O}}(2^{n/3})$ queries to oracle.
- A quantum algorithm that solves FPEP in $\mathcal{O}(n^2)$ memory but $\tilde{\mathcal{O}}(2^{n/2})$ queries.
- A lower bound of $\Omega(2^{n/2})$ for any classical algorithm solving FPEP. Earlier works proved lower bounds for exponential error case, we present a new proof in the constant probability of success regime.
- Exact analyseœs of classical random walk and non-backtracking random walk on welded trees.

Related works

- There has been several efficient quantum algorithms for FEP over the years. An optimal $\mathcal{O}(n)$ query algorithm was shown using multi-dimensional quantum walks [JZ23].
- Gilyen, Hastings, and Vazirani recently used a modified welded tree to prove a sub-exponential separation between classical algorithms and quantum adiabatic computation without a sign problem. [GHV21]
- While Aaronson's question on FPEP remains open, recent results use FEP on welded trees to efficiently find a path in graphs containing welded trees as induced subgraphs [Jia].

Related works

- Ben-David et al. used the welded tree as a gadget to construct a property testing problem for bounded-degree graphs in the adjacency list model, demonstrating an exponential quantum-classical query complexity separation [BDCG+24].
- A preprint by Li shows that sunflower-graphs, which are almost expanders, enable exponential speedup for quantum walks in finding an induced path, outperforming classical algorithms [LT24].

Conclusion

- Is it possible to solve FPEP in $\mathcal{O}(2^{n/c})$ queries with $c > 3$?
- Is the conjecture that any classical algorithm solving FEP in $2^{\mathcal{O}(n)}$ queries must use $2^{\mathcal{O}(n)}$ space true?
- What is the exact quantum lower bound for algorithms in $\mathcal{F}$?
- A generalised quantum lower bound for FPEP?
- Is there a threshold on the number of edges in the weld beyond which the quantum walk algorithm for FEP would require exponential queries?
  - We have some preliminary results for a partially welded tree with even half the weld edges offers a quantum advantage.

Thank you!

References I

Scott Aaronson.
Open problems related to quantum query complexity.
ACM Transactions on Quantum Computing, 2(4):1–9, 2021.

Shalev Ben-David, Andrew M. Childs, András Gilyén, William Kretschmer, Supartha Podder, and Daochen Wang.
Symmetries, graph properties, and quantum speedups.
SIAM Journal on Computing, 53(6):FOCS20–368–FOCS20–415, 2024.

Aleksandrs Belovs.
Global Phase Helps in Quantum Search: Yet Another Look at the Welded Tree Problem.
4 2024.

Gilles Brassard, Peter HØyer, and Alain Tapp.
Quantum cryptanalysis of hash and claw-free functions: Invited paper, page 163–169.
Springer Berlin Heidelberg, 1998.

Andrew M Childs, Richard Cleve, Enrico Deotto, Edward Farhi, Sam Gutmann, and Daniel A Spielman.
Exponential algorithmic speedup by a quantum walk.
In *Proceedings of the thirty-fifth annual ACM symposium on Theory of computing*, pages 59–68, 2003.

Andrew M Childs, Matthew Coudron, and Amin Shiraz Gilani.
Quantum algorithms and the power of forgetting.
arXiv preprint arXiv:2211.12447, 2022.

András Gilyén, Matthew B. Hastings, and Umesh Vazirani.
(sub)exponential advantage of adiabatic quantum computation with no sign problem.
In *Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*, STOC 2021, page 1357–1369, New York, NY, USA, 2021. Association for Computing Machinery.

Lov K. Grover.
A fast quantum mechanical algorithm for database search, 1996.

References II

Li Jianqiang.
Exponential speedup of quantum algorithms for the pathfinding problem.

Stacey Jeffery and Sebastian Zur.
Multidimensional quantum walks.
In *Proceedings of the 55th Annual ACM Symposium on Theory of Computing*, pages 1125–1130, 2023.

Yvan Le Borgne and Shrinidhi Teganahally Sridhara.
Time-Space Tradeoffs in Searching for a Path in Welded Trees, Classically or Quantumly.
working paper or preprint, September 2024.

Guanzhong Li, Lvzhou Li, and Jingquan Luo.
Recovering the original simplicity: succinct and deterministic quantum algorithm for the welded tree problem.
In *Proceedings of the 2024 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 2454–2480. SIAM, 2024.

Jianqiang Li and Yu Tong.
Exponential quantum advantage for pathfinding in regular sunflower graphs, 2024.

Shalev (pseudo).
Reply to the randomized query complexity of the conjoinded trees problem, https://cstheory.stackexchange.com/questions/25279/the-randomized-query-complexity-of-the-conjoined-trees-problem, 2014.

Comparision of current algorithms

Algorithm	Queries	Space
1. Classical algorithm	$\tilde{\Theta}(2^{n/2})$	$\tilde{\mathcal{O}}(2^{n/2})$ bits
2. Quantum Algorithm - 1	$\tilde{\mathcal{O}}(2^{n/2})$	$\tilde{\mathcal{O}}(2^{n/2})$ bits, $\mathcal{O}(n^2)$ qubits
3. Its lower memory versions	$\tilde{\mathcal{O}}(2^{n/2})$	$\tilde{\mathcal{O}}(n^2)$ bits, $\mathcal{O}(n^2)$ qubits
4. Random walk	$\left(\frac{4\times 2^n-3}{2^{n+1}}\right)\cdot 2\cdot(3\cdot 2^n-2)$	$2n$ bits
5. Non-backtracking random walk	$\left(\frac{2\times 2^n-1}{2^{n+1}}\right)\cdot 2\cdot(3\cdot 2^n-2)$	$2n+\lceil \ln_2 C\rceil$ bits
6. Quantum Algorithm - 2	$\tilde{\mathcal{O}}(2^{n/3})$	$\tilde{\mathcal{O}}(2^{n/3})$ bits, $\tilde{\mathcal{O}}(n^2)$ qubits
1. Classical LB 2. Quantum LB	$\tilde{\Omega}(2^{n/2})$ $\Omega(\exp(n))^*$	? *Only for $\mathcal{F}$